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Some Conjectures on Average of Fibonacci and Lucas Sequences

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Abstract

The arithmetic mean of the first n Fibonacci numbers is not an integer for all n. However, for some values of n we can observe that it is an integer. In this paper we consider the sequence of integers n for that the average of the first n Fibonacci numbers is an integer. We prove some interesting properties and present two related conjectures.

Keywords: Fibonacci number, Lucas number, Pisano period, rank of appearance, restricted period.

2020 MSC: 13D45, 39B42.

1. Introduction

The Fibonacci sequence $(F_n)_{n\geqslant 0}$ is defined by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$, A000045 [9]. Fibonacci numbers have been extensively studied [3, 4]. Numerous fascinating properties are known as, for instance, their close relation to the binomial coefficient:

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

The average of the first n terms of the Fibonacci sequence is not always an integer. For instance, for $n = 3$ we have $\dfrac{1+1+2}{3} = \dfrac{4}{3}$, but for $n = 1,\ 2,\ 24,\ 48,\ \ldots$ are $\left(\frac{1}{n}\sum_{i=1}^{n} F_i\right)_{n\geqslant 1}$ integers.

In this paper, we explore the following question: Which terms of the sequence

$$A_F(n) = \left(\frac{1}{n}\sum_{i=1}^{n} F_i\right)_{n\geqslant 1} \tag{1.1}$$

are integers?

We give a characterization for the values of n for that $A_F(n)$ is an integer. Moreover, we present a construction of finding infinitely many n that satisfy the given conditions. Further, we show that there are infinitely many n for that 6 is a divisor of the sum of the first n Fibonacci numbers.

Finally, we show that $A_F(p)$ is not an integer if p is an odd prime number.

Our work is based on the results in [4, 5, 7, 8, 9, 11].

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2. Main results

First, we recall some definitions and important theorems [1, 2, 10]. A fundamental identity that we use in this paper is [4, Theorem 5.1]

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1. \tag{2.1}$$

The Lucas numbers $(L_n)_{n\geqslant 0}$, are defined by the same recurrence relation as the Fibonacci numbers with different initial values (see A000032).

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}, \quad \text{for} \quad n \geqslant 2.$$

The following relations between Fibonacci numbers and Lucas numbers can be found in [4]:

$$F_{4k+1} - 1 = F_{2k}L_{2k+1}, \tag{2.2}$$
$$F_{4k+2} - 1 = F_{2k}L_{2k+2}, \tag{2.3}$$
$$F_{4k+3} - 1 = F_{2k+2}L_{2k+1}, \tag{2.4}$$
$$F_{2k} = F_k L_{2k}. \tag{2.5}$$

An integer a is called a quadratic residue modulo p (with $p > 2$) if $p \nmid a$ and there exists an integer b such that $a \equiv b^2 \pmod{p}$. Otherwise, it is called a non-quadratic residue modulo p.

Let p be an odd prime number. The Legendre symbol is a function of a and p defined as

$$\left(\frac{a}{p}\right) = \begin{cases} +1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1, & \text{if } a \text{ is a non-quadratic residue modulo } p, \\ 0, & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

We note that for a prime number p the Legendre symbol, $\left(\frac{5}{p}\right)$, is equal to

$$\left(\frac{5}{p}\right) = \begin{cases} +1, & \text{if } p \equiv \pm 1 \pmod{5}, \\ 0, & \text{if } p \equiv 0 \pmod{5}, \\ -1, & \text{if } p \equiv \pm 2 \pmod{5}. \end{cases}$$

Consider the sequence of the Fibonacci numbers modulo 8:

$$0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, 2, 3, 5, \ldots$$

We observe that the reduced sequence is periodic.

Lagrange [5] showed that this property is true in general, i.e., that the Fibonacci sequence is periodic modulo m for any positive integers $m > 1$.

Definition 2.1. For a given positive integer m, we call the least integer such that $(F_n, F_{n+1}) \equiv (0, 1) \pmod{m}$ the (Pisano) period of the Fibonacci sequence modulo m and denote it by $\pi(m)$.

The first few values of $\pi(n)$ are given as the sequence A001175 in [9].

We recall as a lemma the fixed point theorem of Fulton and Morris [2].

Lemma 2.2 (Fixed Point Theorem [2]). Let m be a positive integer greater than 1. Then $\pi(m) = m$ if and only if $m = (24)5^{\lambda-1}$ for some $\lambda > 0$.

For instance, with $m = 8$ we have $\pi(8) = 12$ and $\alpha(8) = 6$. The 12 terms in the period form two sets of 6 terms. The terms of the second half are 5 times the corresponding terms in the first half (mod 8). For the Lucas sequence $F_n = U_n(P, Q)$; Robinson [8], we have $t \equiv F_{\alpha(m)-1}(-Q) \pmod m$ is the multiplier between consecutive parts of length $\alpha(m)$ of the period. If the (mod m) order of t is r then $\pi(m) = r\alpha(m)$. Here $F_n = U_n(1, -1), (P, Q) = (1, -1), \alpha(8) = 6, t = 5, r = 2$; thus $\pi(8) = 2 \cdot 6 = 12$; Robinson [8]. The 12 terms in the period form two sets of 6 terms. The terms of the second half are 5 times the corresponding terms in the first half (modulo 8). The next definition is

Definition 2.3. For a given positive integer, we call the least integer such that $(F_n, F_{n+1}) \equiv \sigma(0, 1) \pmod m$ for some positive integer σ the restricted period of the Fibonacci sequence modulo m and denote it by $\alpha(m)$.

Robinson [8] showed the following theorems.

Theorem 2.4. i) $m \mid F_n$ if and only if $\alpha(m) \mid n$, and

ii) $m \mid F_n$ and $m \mid F_{n+1} - 1$ if and only if $\pi(m) \mid n$.

Theorem 2.5. If p is a prime, then

i) $\alpha(p) \mid (p - \left(\frac{5}{p}\right))$,

ii) if $p \equiv \pm 1 \pmod 5$, then $\pi(p) \mid (p - 1)$, and

iii) if $p \equiv \pm 2 \pmod 5$, then $\pi(p) \mid 2(p + 1)$.

The exponent of the multiplier of the Fibonacci sequence modulo p, $t \equiv F_{\alpha(p)-1} \pmod p$ is $\frac{\pi(p)}{\alpha(p)}$ and can only take the values 1, 2 and 4.

For a positive integer n and a prime p, the p-adic valuation of n, $\nu_p(n)$, is the exponent of the highest power of p that divides n.

Legendre's classical formula for the p-adic valuation of the factorials is well known:

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor .$$

We recall Lengyel's lemma [6] about the p-adic evaluation of Fibonacci numbers in cases $p = 2, 3$ and 5.

Lemma 2.6 ([6], Lemmas 1 and 2). For all $n \geqslant 0$, we have $\nu_5(F_n) = \nu_5(n)$. On the other hand,

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 1, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

and

$$\nu_3(F_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod 4; \\ \nu_3(n) + 1 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

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